Let's start with ($\tau_{in} = 0$)

$$\tilde{R}(\eta, \tau) \; = \; \tilde{r}^{(0)} \tau + \frac{\tilde{\alpha}_s}{2} \int_0^{\tau - s(\eta-\eta')\theta(\eta-\eta')} d\tau' \int_0^\infty d\eta' \, \tilde{R}(\eta', \tau') \tag{1}$$

which can be rewritten as

$$\tilde{R}(\eta, \tau) \; = \; \tilde{r}^{(0)} \tau + \frac{\tilde{\alpha}_s}{2} \int_{\max\{0, \eta - \tau/s\}}^{\eta} d\eta' \int_0^{\tau - s\eta + s\eta'} d\tau' \, \tilde{R}(\eta', \tau') + \frac{\tilde{\alpha}_s}{2} \int_\eta^\infty d\eta' \int_0^\tau d\tau' \, \tilde{R}(\eta', \tau') \tag{2}$$

Differentiating Eq. (2) wrt η (a la Kazu) we get

$$\frac{\partial \tilde{R}(\eta, \tau)}{\partial \eta} \; = \; - \frac{\tilde{\alpha}_s s}{2} \int_{\max\{0, \eta - \tau/s\}}^{\eta} d\eta' \, \tilde{R}(\eta', \tau - s\eta + s\eta'). \tag{3}$$

Differentiating Eq. (2) wrt τ we get

$$\frac{\partial \tilde{R}(\eta, \tau)}{\partial \tau} \; = \; \tilde{r}^{(0)} + \frac{\tilde{\alpha}_s}{2} \int_{\max\{0, \eta - \tau/s\}}^{\eta} d\eta' \, \tilde{R}(\eta', \tau - s\eta + s\eta') + \frac{\tilde{\alpha}_s}{2} \int_\eta^\infty d\eta' \tilde{R}(\eta', \tau). \tag{4}$$

Combining Eq. (3) with Eq. (4) we end up with

$$\frac{\partial \tilde{R}(\eta, \tau)}{\partial \tau} \; = \; \tilde{r}^{(0)} - \frac{1}{s} \frac{\partial \tilde{R}(\eta, \tau)}{\partial \eta} + \frac{\tilde{\alpha}_s}{2} \int_\eta^\infty d\eta' \tilde{R}(\eta', \tau). \tag{5}$$

Differentiating Eq. (5) wrt η we get

$$\frac{\partial^2 \tilde{R}(\eta, \tau)}{\partial \tau \, \partial \eta} \; = \; - \frac{1}{s} \frac{\partial^2 \tilde{R}(\eta, \tau)}{\partial \eta^2} - \frac{\tilde{\alpha}_s}{2} \tilde{R}(\eta, \tau) \tag{6}$$

with the initial conditions given by

$$\tilde{R}(\eta, \tau = 0) \; = \; 0 \tag{7}$$

and

$$\left. \frac{\partial \tilde{R}(\eta, \tau)}{\partial \tau} \right|_{\tau = 0} \; = \; \tilde{r}^{(0)} \tag{8}$$

as can be derived from Eq. (2) and Eq. (4). Now we have a well-defined PDE, the way Derek likes it. Right away one can see that factorization ansatz Derek is proposing while being able to solve Eq. (6) would never satisfy initial condition of Eq. (8). Factorization Derek sees in his numerical solution is probably due to large values of τ he's using for the asymptotics... don't know.

Anyway, the solution I found is

$$\tilde{R}(\eta, \tau) \; = \; \tilde{r}^{(0)} \sqrt{\frac{2\,\tau}{\tilde{\alpha}_s (\tau/s - \eta)}} \, I_1(\sqrt{2 \tilde{\alpha}_s \tau (\tau/s - \eta)}), \tag{9}$$

which satisfies Eq. (6) together with Eq. (7) and Eq. (8) and must be unique.